EMERGENCE OF GEOMETRY AND CONCEPTUAL CHANGES IN THEORY OF RATIOS IN THEORETICAL MUSIC IN THE 16TH CENTURY

Oscar João ABDOUNUR
Instituto de Matemática
Universidade de São Paulo, Brazil

ABSTRACT — The end of the 15th and the beginning of the 16th century witnessed intense structural changes in the conceptions underlying ratios and proportions as well as a significant extension in the spectrum of techniques used in theoretical music. In this context, changes in the conception of ratio brought about the strengthening of the arithmetization of theories of ratio and also the use of geometry as an instrument for solving structural problems in theoretical music. The sixteenth century represented thus a major revolution in the production of treatises on theoretical music and saw, in distinction to the Pythagorean tradition, the introduction of geometry as a tool not only to solve problems such as the division of the tone but also to solve theoretical problems related to the systematization of the temperament, symbolizing a substantial conceptual change in the mathematical foundations of theoretical music and eventually in the theory of ratios.

The experiment of Pythagoras: compounding ratios

Mathematics and music were already linked in Antiquity. In the so-called experiment of the monochord, Pythagoras is credited with having established the correspondence between musical intervals and ratios of a string, discovering that certain intervals could be produced by dividing the string in simple ratios *a:b* such that *b* represented the whole string whereas *a* represented a part of the string. In particular, the intervals of the octave, fifth and fourth were produced by simple ratios 1:2; 2:3 and 3:4, respectively. These intervals were called perfect consonances, and the Pythagorean consonances consisted strictly of the intervals whose underlying ratios were formed only by the small numbers 1,2,3 and 4 – the *Tetraktys*.

Inquiries related to the interrelationships between Greek music and the development of pure mathematics were already conducted in the beginning of the 20th century by P. Tannery (Tannery, 1915) and in the seventies by Szabo, who also raised similar questions in the attempt to show that pre-Eudoxan theory of proportions developed initially as an inheritance from the Pythagorean theory of music (Szabo, 1978). In the latter, some indicators of such an inheritance are found in connection with issues such as Euclid's constraint on the operation of *compounding ratios* implied yet not explicitly defined, for instance, in proposition 23, Book VI (Heath, 1956, 247).

Equal division of the tone and theories of ratios underlying music theory

In the context of the discussions on musical theory having ratios as their basis, the problem of the equal and proportional numerical division of the whole tone interval sounding between strings with the length ratio of 9:8 occupied the minds of the theorists from Antiquity up to Renaissance. It would play an important part in the historical process that led to the emergence of equal temperament. Attempts to divide the tone were already made in Antiquity, for instance by Aristoxenus (fourth century B.C.), who conceived the theoretical nature of music as essentially geometric, understanding pitches, musical intervals and also distance as unidimensional magnitudes – continuous quantities – that should follow the rules of the Euclidean geometry and should be capable of being divided continuously, which inevitably raises questions concerning the nature of ratio in this context. In contrast with the Pythagoreans, who defended the position that musical intervals could properly be measured and expressed only as mathematical ratios involving whole numbers, Aristoxenus rejected this position, asserting instead that the ear was the sole criterion of musical phenomena (Winnington-Ingram 1995, 592).

The problem of the division of the tone arose from the Pythagorean discovery of numerical indivisibility of a superparticular integer ratio (a ratio reducible to an expression of the type (n+1):n) by its geometrical mean, in the case, the ratio 9:8. Given three integers $A < x < B$, where the ratio A:B is superparticular, x cannot be both an integer and at the same time fulfill the condition A:x=x:B, it means, it can not be the geometric mean between A and B. Mathematically, the equal division of the tone 8:9 provides incommensurable ratios underlying musical intervals, a result whose proof belongs to the body of Pythagorean knowledge.

In preferring geometry to arithmetic in solving problems involving relations between musical pitches, Aristoxenus sustained as mentioned before and also against the Pythagoreans, the possibility of dividing the tone into two equal parts, conceiving musical intervals – and indirectly ratios – as one-dimensional and continuous magnitudes. This idea provoked a large number of reactions expressed for instance in the *Sectio Canonis* (Barbera 1991, 125), which was in Antiquity attributed to Euclid and much later in the *De institutione musica* (Bower and Palisca 1989, 88) of Boethius in the early Middle Ages, which gave birth to a strong Pythagorean tradition in theoretical music throughout the Middle Ages. Following the Platonic-Pythagorean tradition, a great part of medieval musical theorists sustained the impossibility of the equal division of the tone, which would mathematically lead to incommensurable ratios underlying musical intervals. Gradually, the need to carry out the temperament gave birth to different attempts to divide the tone.

Goldman suggests that Nicholas Cusanus (1401-1464) seems to be the first to assert in *Idiota de Mente* that the musical half-tone is derived by *geometric division* of the whole-tone, and hence would be defined by an irrational number (Goldman, 1989, 308). As a consequence, Cusanus would be the first to formulate a concept that set the foundation for the equal temperament proposed in the work of the High Renaissance music theorists Faber Stapulensis (1455-1537) and Franchino Gafurius (1451-1524), published half a century later (Goldman, 1989, 308). Nevertheless, one can find in the Byzantine tradition Michael Psellus (1018-1078), who suggested in his *Liber de quatuor mathematicis scientijs, arithmetica, musica, geometria, [et] astronomia* (Psellus, 1556) a geometrical division of the tone, whose underlying conception implies an understanding of ratio as a continuous magnitude. Also concerning the division of the tone before Cusanus, Marchetus of Padua (1274 ? --?) proposed, in his *Lucidarium in Arte Musice Planae* written in 1317/1318, the division of the tone into five equal parts (Herlinger, 1981, 193), an innovation of extraordinary interest which made Marchettus the first in the Latin tradition to propose such a division, but without any mathematical approach.

In the context of recovery of the interest in ancient texts and of the use of geometry to solve problems in theoretical music at the end of the fifteenth century and the beginning of the sixteenth century, the Bohemian mathematician and music theorist Erasmus of Höritz emerged as one of the German humanists most articulate with musical matters. Erasmus went back to the Greek sources of the doctrines of Boethius, communicating to musical readers an important fruit of the revival of interest in ancient texts. He wrote his *Musica*, where he suggested in chapter seventeen a equal division of superparticular ratios (Erasmus Horicius, 1500?, fo. 66^{v}).

Erasmus stated that any part of any superparticular ratio could be obtained, in particular the half of 8:9, which corresponds to divide equally the whole tone. In chapter seventeen of his *Musica*, he established a numerical procedure which would lead to the geometrical mean between the terms of ratio 9:8 underlying the tone. He attempted to arrive to an expression of the ratio for the supposedly equally proportional halves of the whole tone interval using very large integers numbers. He did it first by constructing assured by proposition 18 of Book VII of the Elements, which asserts, that a:b :: am:bm. Following his method, the half of the 9:8 ratio of a tone could be obtained by the geometric mean of its expansion into the term 34828517376: 30958682112, which is nothing but (3869835264 x 9):(3869835264 x 8), that is, he applied proposition 18 mentioned above to a:b and m equal to 9:8 and 3869835264, respectively. In this case, since decimal fractions were not available at this time, he tried to get precision through large integer numbers rather than with places after decimal point, proposing an abstract numerical method for the given problem.

The Pythagorean-Euclidean proof of the indivisibility of a superparticular ratio is naturally irrefutable in the domain of rational number. Nevertheless, in his *musica*, Erasmus asserts that "*... in musical demonstrations we are forced to use all kinds of ratios... since not all shapes of consonances and also dissonances are founded in rational ratios and for that reason we must not neglect the ratios of surds...*" (Erasmus Horicius, 1500?, fo. 61^{v}). Erasmus proposed here nothing less than a consideration of incommensurable ratios in musical contexts. For music theoretical purposes, in order to make use of Eudoxus' theory of book V of Euclid's Elements on which theory of ratios of surds is based and which deals with abstract quantities with continuous nature, he established a link between continuous and discrete quantity. Erasmus realized that the sought for a geometrical mean to the ratio underlying the whole tone could not result in a rational number and instead of changing the domain at this point from discrete quantity of numbers to continuous quantity of geometrical lines, he proposed a number continuum, although not explicitly, creating a very dense discrete point space between the original terms 9 and 8 by their expansion, which also did not allow him to locate the precise mean. If he really thought that he was capable of dividing the sesquioctave ratio in terms of a purely numerical operation, he must have possessed an at least rudimentary concept of number continuum, an assumption which is corroborated in a passage later on in chapter seventeenth in which he seemed to refer directly to the idea of such a continuum mentioning Boethius as a prisoner of the Pythagorean doctrine of discrete integer number not having access to all ratios of numbers, which followed each other continuously in the manner of continuous quantities (Erasmus Horicius, 1500?, fo. 67^{v}).

Theoretically based on many geometrical propositions and, unusually, modeled on Euclidean style, his *Musica* dealt with ratio as a continuous quantity, announcing perhaps what would emerge as a geometric tradition in the treatment of ratios in theoretical music contexts during the sixteenth century. Interestingly, Erasmus could have easily solved the equal division of the tone making use of the proposition of Euclid's Elements which provide the geometrical mean through the high of a rectangle triangle. Nevertheless, he tried, although missing the concept of infinity, to make use of a numerical method to approach such a mean, which also indicates his consideration with Pythagorean conceptions in theoretical music contexts.

The change from an arithmetical to a geometrical basis in the theory of music represents a meaningful structural transformation in the basis of theoretical music, strongly tied to the change in the conception of western music mentioned before.

The discussions concerning the division of the tone and consequently underlying conceptions of ratios in music and mathematics contexts are naturally linked to the structures of theories of ratio in such contexts. Up to the Renaissance, the treatment of ratios had no clear and well-defined structure. Some of the traditions had mainly arithmetical features, others geometrical and musical ones, whereas still others incorporated both of these tendencies. Sylla discussed the confusion over the geometrical and the arithmetical traditions of ratios, showing how the two traditions in the context of compounding and multiplying were "strangely mingled" (Sylla, 1984). She categorizes two traditions within the Greek and medieval treatment of ratios, one associated with theoretical mathematics, with music, and with physics, particularly found in Bradwardine's *De proportionibus*; and the second associated with practical calculations using ratios and with astronomy (Sylla, 1984, 11). She argues that "These two traditions may not encompass all ancient and medieval concepts of ratio. Neither were these traditions always separate -- in fact, they were often strangely mingled. Nevertheless, they represent two poles of the ways in which ratios and the operations on ratios could be treated" (Sylla, 1984, 17). Such different structures, which had kept up with the concepts of ratio and proportion since Antiquity, shaped distinct theories for such concepts as underlay treatises of mathematics and of music up to the Renaissance.

The mathematical basis of Renaissance theoretical music: from arithmetic to geometry

The period from the end of the fifteenth century to the end of sixteenth century witnessed more intense structural changes in the conceptions underlying ratios and proportions in the contexts of theoretical music. With the need of equal temperament which brings together the need of the equal division of the whole tone and consequently structural changes in the conceptions of ratios, treatments with such concepts in theoretical music ceased to be a subject exclusively of arithmetic and became a subject of geometry.

In this context, Erasmus Horicius contributed immensely to the introduction of geometry as an instrument for solving structural problems in theoretical music. Notwithstanding the announcements of the need for geometry in theoretical music by previous authors, Erasmus could be considered the first in the Renaissance to apply Euclidean geometry extensively in his *Musica* (Erasmus Horicius, 1500?) for the resolution of structural problems in theoretical music. Relying mainly on books V and VI of Euclid, Horicius used geometry in different ways to solve musical problems, applying it to intervals, in contradiction to the Boethian arithmetical tradition. He used in his *Musica* the *denominatio* terminology taken from Campanus's Latin translation of *the Elements*, a procedure which also contributed to the emergence of ratio as a continuous quantity and consequently to an arithmetical theory of ratio in the context of theoretical music, together with the considerations mentioned before. Making use of geometrical resources hitherto unusual in musical contexts, Erasmus showed that the intervals of the fifth (3:2) and the whole tone could be divided through a proportional mean, namely by finding a magnitude b between a and c so that $a{:}b$ is proportional to $b{:}c$ considering the whole tone mathematically expressed by $a{:}b$, although such resources involved potentially irrational numbers. Erasmus represents an intensification in the conceptual change undergone by theoretical music at this time, and his contribution is relevant to the research on mathematics and music at the end of the fifteenth century and beginning of the sixteenth century at the University of Paris, inasmuch as one can find the use of geometry in the solution of musical problems, for instance in the geometric division of superparticular intervals presented in Faber Stapulensis's *Elementalia musica*, first published in 1494. This work had influence in the Spanish tradition of theoretical music in the sixteenth century, with authors like Pedro Ciruelo (1470-1548) and Juan Bermudo (1510-1565), who also presented respectively in the works *Cursus quatuor mathematicarum artium liberalium*, published in Alcalá de Henares in 1516 (Ciruelo, 1526) and *Declaración de Instrumentos,* published in 1555 in Osuna (Santiago Kastner,1957) the same division of the tone with the geometrical mean presented by Faber Stapulensis. In the Iberian Peninsula, the tendency to use geometry occurred also in Salinas's *De Musica* published in Salamanca in 1577, which contains a geometrical systematization for the equal Temperament that makes extensive use of Euclid's *Elements*.

Such a tendency spread also to the German and Italian production in theoretical music. For instance, the German mathematician Heinrich Schreiber (1492-1525) published in the appendix *Arithmetica applicirt oder gezogen auff die edel kunst Musica* of his "Ayn new kunstlich Buech…" of 1521 (Bywater, 1980) a geometric division of the tone into two equal parts making, use of the Euclidean method for finding the geometric mean. He also operated with ratios with a very arithmetical structure, for instance, compounding them as one, anachronistically, multiplies fractions.

In the Italian tradition the tendency to use geometry was also strong. A representative example of such a tendency is Gioseffo Zarlino, a leading Italian theorist and composer in the sixteenth century. One of the most important works in the history of music theory, Zarlino's *Le istitutioni harmoniche* (1558), represents an important attempt to unite speculative theory with the practice of composition on the grounds that "music considered in its ultimate perfection contains these two parts so closely joined that one cannot be separated from the other (Palisca, 1995, 646). The tendencies for reconciling theory and practice also manifested themselves in this period in the context of structural problems underlying theoretical music. Such a reconciliation seemed to be incompatible with a Pythagorean perspective on theoretical music, in which there was no place for geometry, an essential tool for modeling a new language claimed by practical music.

In this context, it is worthwhile to mention Zarlino's *Sopplimenti musicali* (1588), in which the Italian theorist demonstrated much greater penetration into the ancient authors, particularly Aristoxenus and Ptolemy, than in *Le istitutioni harmoniche* (Palisca, 1995, 648). In spite of the still existing authority of Pythagoreanism in the context of theoretical music in the sixteenth century, Zarlino's *Sopplimenti musicali* already gave evidence of the tension between speculative theory and practice in the contexts of structural problems in theoretical music, inasmuch as it presented geometrical solutions for the equal temperament but was also based on Pythagorean foundations.

Interestingly, in the last page of the *Sopplimenti musicali*, that is, at the end of chapter 32 of book 8 of the third volume, Zarlino wrote "... *che la Musica più tosto sia sottoposta alla Geometria, che alla Arithmetica*…" (Zarlino, 1588, 330), which means that music is subordinate to geometry rather than to arithmetic. Zarlino published the *Sopplimenti musicali* just before his death in February of 1590. This passage in his last work seems to be the first time that Zarlino assumed explicitly that geometry was not just a theoretical tool together with arithmetic for dealing with problems in theoretical music, but rather constituted a better tool for this task then arithmetic.

Concluding remarks

The sixteenth century represented a major revolution in the production of treatises on theoretical music. It witnessed, in distinction to the Pythagorean tradition, the introduction of geometry as a tool not only to solve the problem of division of the tone but also to solve theoretical problems related to the systematization of the temperament, symbolizing a substantial change in the structures and conceptions of ratio underlying theoretical music and thus in the foundations of theoretical music.

Procedures in theoretic musical contexts like those mentioned above made by Erasmus Horicius and others after him brought to music changes in theories of ratios in direction of conceiving such concepts as continuous quantities and compounding as multiplication, intensifying the conflicts associated with the Pythagorean tradition concerning theoretical music, according to which only whole numbers and ratios of whole numbers – discrete quantities – could serve as the basis for theoretical music, whether through a stiff distinction between consonance and dissonance defined by the first four numbers or through the search for a perfect system of intonation based on commensurable ratios.

Such aspects are representative, in a wider sense, of a change undergone by ratios, compounding ratios and in a wider sense by theories of ratio as well as by the foundations of theoretical music in this period, which gradually ceased to be based exclusively on an arithmetical dogmatism to comprise geometry and experimental principles in its basis.

BIBLIOGRAPHY

BARBERA, Andre (1991) *The Euclidean Division of the Canon*, Lincoln: University of Nebraska Press.

BOWER, Calvin M. and Palisca, Claude Victor (1989) *Fundamentals of Music. Anicius Manlius Severinus Boethius*, New Haven & London: Yale University Press.

BYWATER, Michael F. (1980). *Heinrich Schreiber. Ayn new Kunstlich Buech...1518*, London: Scolar Press.

CIRUELO, Pedro (1526) *Cursus quatuor mathematicarum artium liberalium: quas recollegit atque correxit magister Petro Ciruelus*, Alcalá de Henares: Miguel de Eguía.

COHEN, H. Floris (1984) *Quantifying music. The science of music at the first stage of the Scientific Revolution, 1580-1650*, Dordrecht: D. Reidel Publishing Company.

ERASMUS HORICUS (1500?) *Musica,* Vatican Library, MS Regina lat. 1245.

GOLDMAN, David Paul (1989) Nicholas Cusanus' contribution to music theory, *Rivista Internazionale di musica sacra,* 10/3-4, 308-338.

HEATH, Thomas L. (1956) *The thirteen books of Euclid's elements: translated from the text of Heiberg*, vol. 2: Books III – IX, New York: Dover.

HERLINGER, Jan W. (1981) Marchetto's division of the whole tone, *Journal of the American Musicological Society,* 34,193-216.

MENGE, Henricus (1916) *Euclidis opera omnia viii*, Leipzig: Teubner.

PALISCA, Claude Victor (1995) Zarlino, Gioseffo, in: Sadie, Stanley, The New Grove dictionary of music and musicians, London: Macmillan, 646-649.

PSELLUS, Michael (1556) *Pselli, doctiss. uiri, perspicvvs liber de quatuor mathematicis scientijs, arithmetica, musica, geometria, [et] astronomia: Graece et latine nunc primùm editus*, Basileae: Oporinus.

SANTIAGO KASTNER, Macario (1957) *Fray Juan Bermudo. Declaración de Instrumentos musicales*, Bärenreiter: Verlag Kassel und Basel.

SYLLA, Edith Dudley (1984) Compounding ratios. Bradwardine, Oresme, and the first edition of Newton's Principia, in: Mendelsohn, E. *Transformation and tradition in the sciences. Essays in honor of I. Bernard Cohen*, Cambridge: Cambridge University Press, 11-43.

SCRIBA, Christoph J. (1991) Zum historischen Verhältnis von Mathematik und Musik, *Braunschweigische Wissenschaftliche Gesellschaft, Jahrbuch,* 115-152.

SZABO, Árpád (1978) *The Beginnings of Greek Mathematics*, Budapest: Akademiai Kiado.

TANNERY, Paul (1915) Du Rôle De La Musique Grecque Dans Le Développement De La Mathématique Pure, *Memoires Scientifiques*, Paris, 68-87.

WINNINGTON-INGRAM, R. P. (1995) *Aristonexus*, in: *Sadie, Stanley, The New Grove dictionary of music and musicians*, London: Macmillan, 592.

ZARLINO, Gioseffo (1588) *Sopplimenti musicali del rev. M. Gioseffo Zarlino da Chioggia*, Venetia: Francesco de Franceschi.